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MATH H324 College Geometry

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Honors Research Paper

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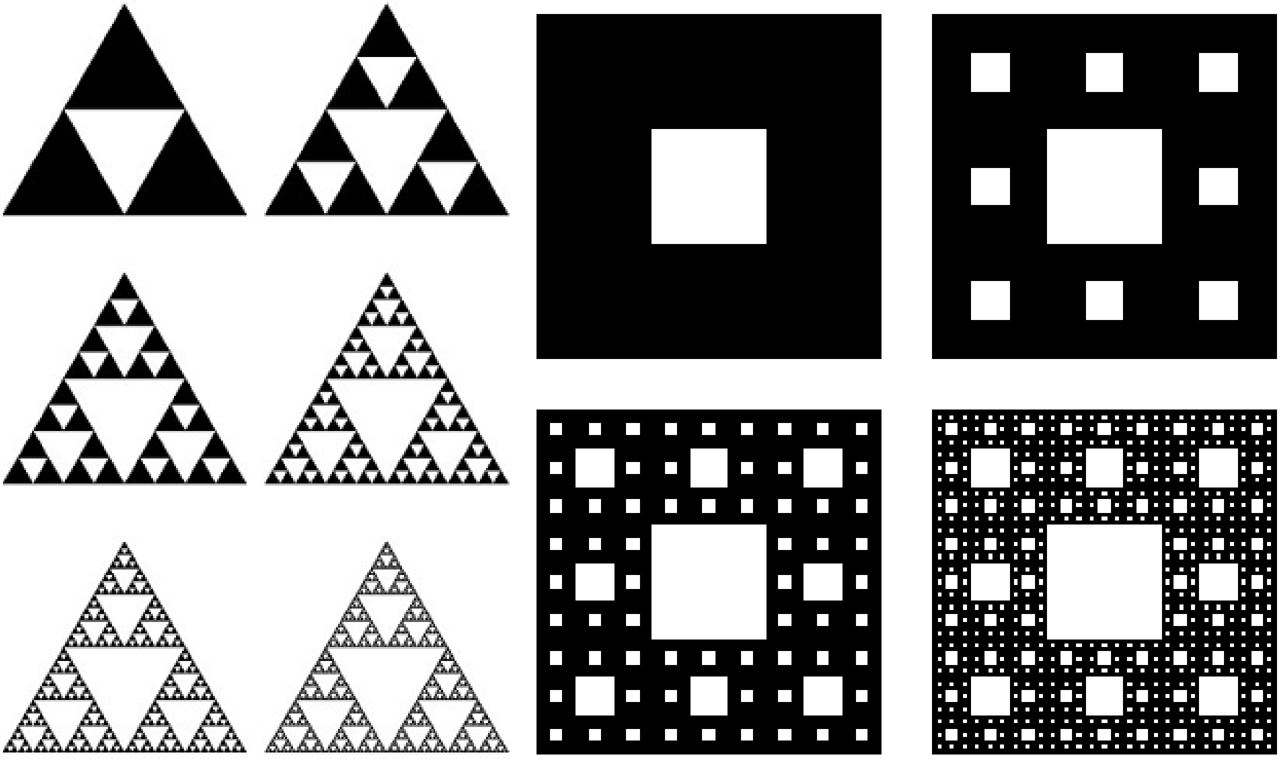
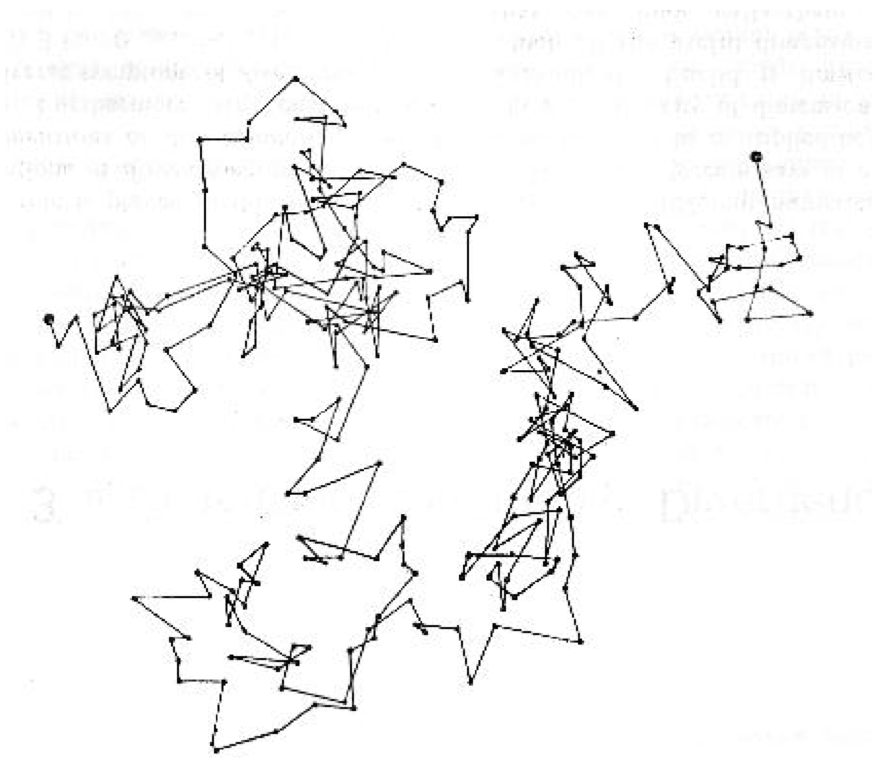
**Fractal Geometry: History and Theory**

Classical Euclidean geometry cannot accurately represent the natural world; fractal geometry is the geometry of nature. Fractal geometry can be described as an extension of Euclidean geometry and can create concrete models of the various physical structures within nature. In short, fractal geometry and fractals are characterized by self-similarity and recursion, which entails scaling patterns, patterns within patterns, and symmetry across every scale. Benoit Mandelbrot, the “father” of fractal geometry, coined the term “fractal,” in the1970s, from the Latin “Fractus” (broken), to describe these infinitely complex scaling shapes. Yet, the basic ideas behind fractals were explored as far back as the seventeenth century; however, the oldest fractal is considered to be the nineteenth century's Cantor set. A variety of mathematical curiosities, “pathological monsters,” like the Cantor set, the Julia set, and the Peano space-filling curve, upset 19th century standards and confused mathematicians, who believed this demonstrated the ability to push mathematics out of the realm of the natural world. It was not until fractal geometry was developed in the 1970s that what past mathematicians thought to be unnatural was shown to be truly representative of natural phenomena. In the late 1970s and throughout the 1980s, fractal geometry captured world-wide interest, even among non-mathematicians (probably due to the fact fractals make for pretty pictures), and was a popular topic- conference sprung up and Mandelbrot's treatise on fractal geometry, *The Fractal Geometry of Nature*, was well-read even outside of the math community. Basically any form in nature can be described mathematically with fractals, and fractals offer a way to translate the natural world into mathematics. This paper will focus on a brief overview of the intricate history of fractal geometry and will lightly touch upon the mathematics behind fractals.



**Sierpinski Triangle (D = 1.5849) and Carpet (D = 1.8928), two early fractals**

Man-made geometric structures were the regular shapes and forms that mathematicians had concerned themselves with for thousands of years. Irregular shapes were at odds with classical mathematics; forms of nature existed outside of the world of mathematics. The nineteenth century saw huge extensions of the idea of symmetry and congruence, which drew from Euclidean and newly discovered non-Euclidean geometry. German mathematician Felix Klein, in 1872, suggested viewing geometry as “the study of the properties of a space which are invariant under a given group of transformations” and that geometry needed to include not only classic shapes but also the yet unnamed fractals and even movements (for example, Brownian motion, which is a “natural fractal”).[1] Klein played with ideas of similar/symmetric objects and transformations/movements in the plane and found that studying the features of the object that were left unchanged by the transformations was the important idea.[2] Klein described symmetry as a balance created by similar repetitions.[3] Iterating the same motion over and over again a number of times creates a pattern or an object that is symmetrical with respect to motion; the individual points of the figure change position but the pattern or the shape of the object, as a whole, remains unchanged.[4]

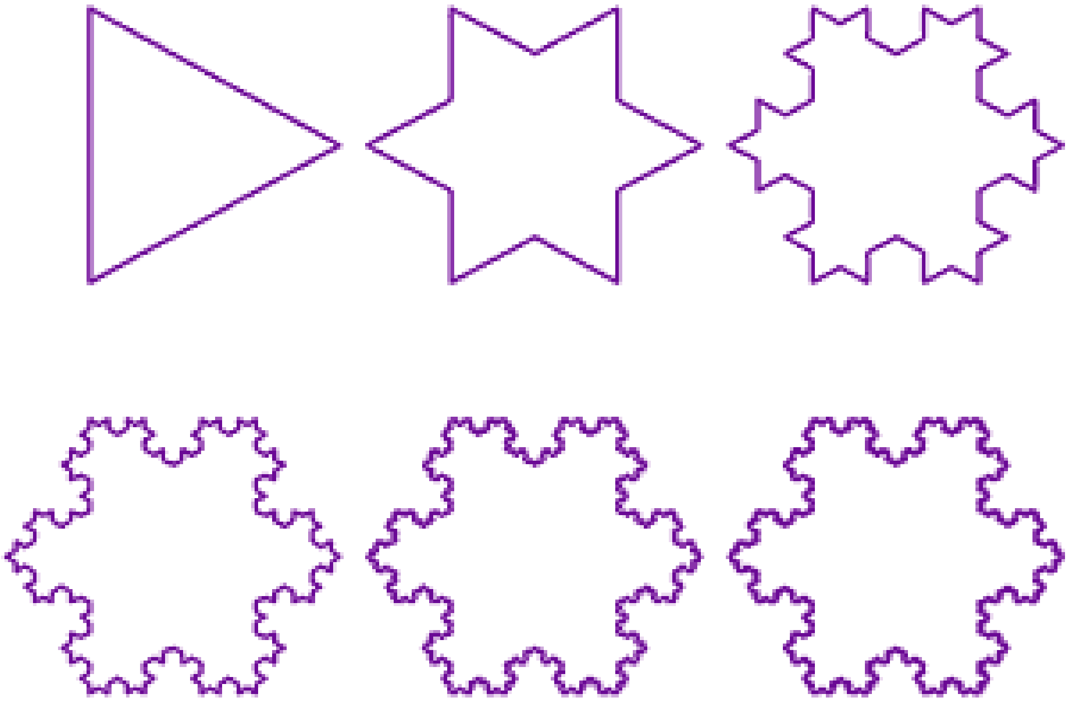


**Brownian Motion**

The so-called "pathological monsters," named so because they made no sense to nineteenth century mathematicians, did not fit in with the definitions of what a curve should be. They could not be classified as classic lines or shapes, and they represented objects that were mathematically infinite. The Cantor set is considered to be the earliest fractal, though the Apollonian gasket, constructed by a simple geometric procedure, dates back to the ancient Greeks. Discovered in 1883 by George Cantor, his set is comprised of infinitely many points and a simple self-similarity. The "paradoxical qualities" of figures like the Cantor set disturbed nineteenth century mathematicians. Things like this had no place in geometry of the past two thousand years.[5] Presently, any fractal dust formed by "roughly self-similar repetition" is considered a Cantor set. The Cantor set begins with an interval between 0 and 1 on a line, the middle third of each successive segment is removed, this is repeated again and again, and soon, many, many, many smaller line segments are created, twice as many segments at each stage, with each segment one third of the length from which it came from. The fractal dust is the set of points that is left behind and not removed. The Cantor set is, basically, the complete set mapped, by one transformation, onto a part of itself. A contemporary mathematician of Cantor would suggest that the set has a dimension of zero but in fact its dimension is about 0.6309. This set, especially, bothered many turn of the century mathematicians. However, it is an essential ingredient for many fractals.

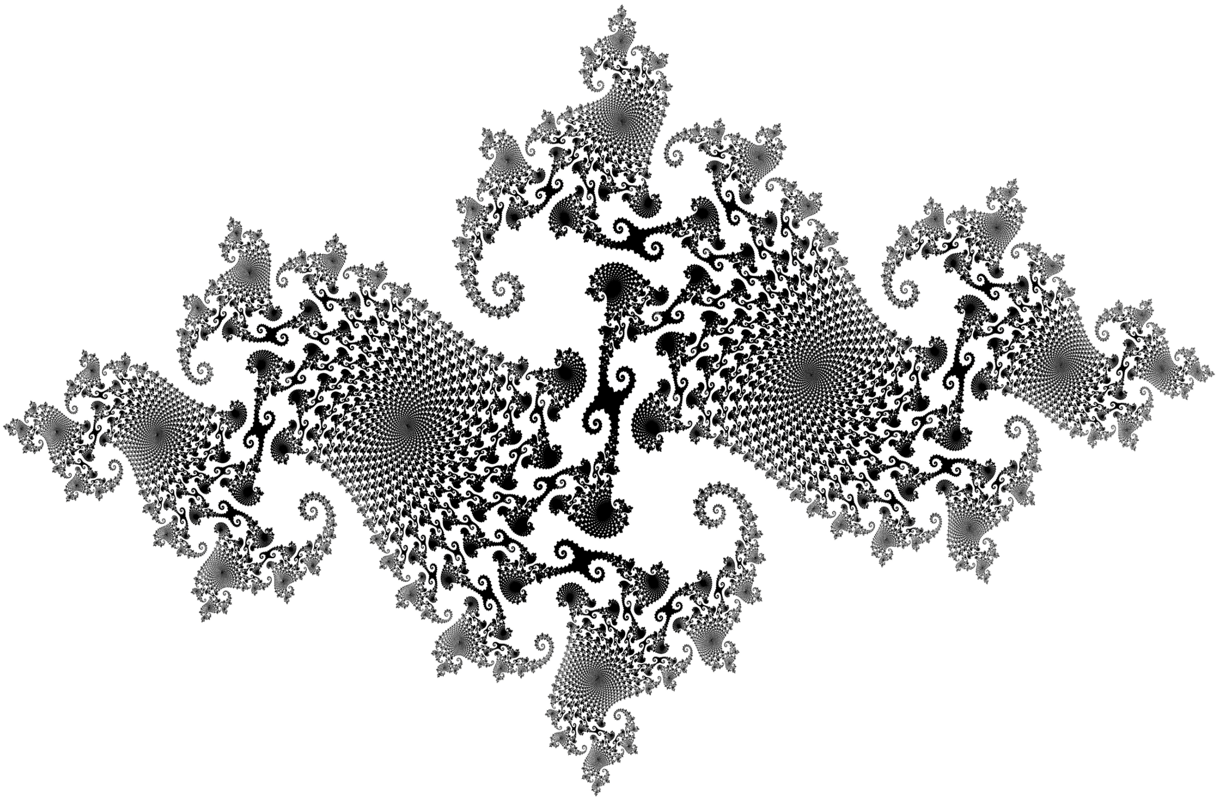
**The Cantor Set**

The Koch curve, another pathological monster, is more than a line but less than a plane, and it is a curve without any tangents; it is a continuous loop.  It was discovered by Helge von Koch in 1904. Every part is itself a miniature copy of the whole and is full of Cantor point-sets. The Koch snowflake is a figure of finite area but is infinitely long (infinite length in finite space). If you drew a circle around the Koch snowflake, the curve would never cross it. The Koch curve helped highlight the problem of defining the length of a coastline. A coastline is infinitely long, just like the Koch curve; you can always find finer and finer indentations to measure. Benoit Mandelbrot addressed the problem of measuring a coastline later on in his paper, “How Long is the Coastline of Britain?,” in which Mandelbrot suggested that while you could not measure the actual length, as it is infinite, you could measure its “roughness,” which required rethinking the notion of dimension. He proposed that the rougher something is, the higher the dimension it has. A coastline has a dimension greater than 1 (and less than 2), but each coastline has a different fractal dimension. Fractal dimension will be discussed later within this paper. The monsters, as stated before, disturbed the turn of the century mathematicians; they were disrespectful to all reasonable intuition about shapes and forms. However, though they believed many of these were mere curiosities without practical value, now such “pathological curves” occur everywhere in pure and applied mathematics.



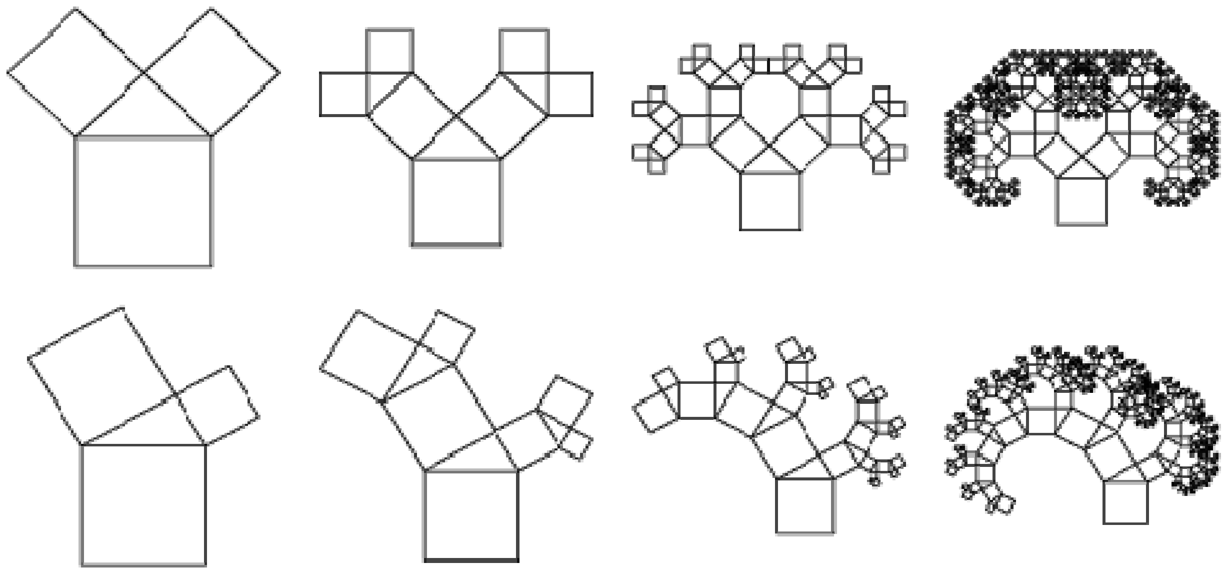
**The Koch Snowflake**

Gaston Julia, in the early twentieth century, studied what happens when you take an equation, you put a number into the equation, and then, you take the number you get and put that back into the equation, and what happens if you keep on iterating the same process. What one ends up with is a set. However, you cannot compute or know the whole set. Julia knew his sets could not be described with words or any concepts found within Euclidean geometry. Julia's work contained no images; his elaborate sets only existed in his mind. The contents of his work were largely ignored for half a century until the advent of the computer made everything possible. Julia fractals involve iterative mappings involving complex numbers and functions, but with the property that all angles remain the same.[6] These fractals occur often in mathematics. There are two types of Julia fractals- wholly connected or wholly disconnected, where the former is made up of infinite separate points (Cantor). If connected, the fractal is a succession of lines, like a single closed curve, loops with loops with loops, or a dendrite.[7] The Julia sets, much like the nineteenth century monsters, demanded the use of fast computers. Computers make iteration easy. The extent of the Julia set and monsters would not be known until Mandelbrot started his own work on what would go onto to be named fractal geometry.



**An Example of a Julia Set, D = 2**

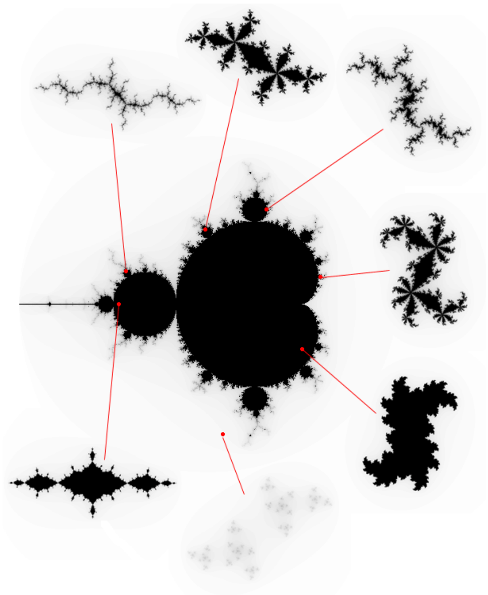
Pythagoras studied the figure in which squares were placed on sides of a right triangle and proved that the sum of the areas of the squares on two sides would equal that of the square on the hypotenuse; this figure has since grew into the Pythagoras tree. In 1957 A.E. Bosman wrote a book on the geometrical shapes found in nature. He relied on the spiral motif found in figures, like shells. Spirals have become the building blocks of fractals and can be thought of a the building blocks of the natural world.[8] Bosman also created the Pythagoras tree in 1942, which is full of spirals and is a “fine example of a mathematical fractal."[9] The lopsided Pythagoras tree is a generalization in which a square is followed by an arbitrary right triangle; it's a curled shape is determined by a similarity transformation. The dimension of both trees is 2.



**The Pythagoras Tree and Loop-sided Pythagoras Tree**

As early as 1960, Mandelbrot had an inkling about the existence of fractals. He had dabbled in economics and found an odd pattern in cotton prices, after he visited a Harvard professor, who could not get the prices changes to fit the bell curve. Mandelbrot, instead, looked for patterns across every scale and symmetry among both the large and small scales. When he plugged the prices into a computer, he found that though each price was random and unpredictable, the sequence of changes was independent of scale, and that the curves for daily price changes matched those of monthly price changes- there was an unexpected kind of order.[10] Instead of separating the tiny changes from the big ones, Mandelbrot bound them together in patterns across every scale.[11] His next early encounter with fractals was studying noise error. IBM engineers were concerned with noise in telephone lines. The transmission noise was random, but came in clusters, and the more closely one looked at the clusters the more complicated the patterns of errors seemed. Mandelbrot found that on scales of an hour or a second the proportion of error-free periods to error-ridden periods remained constant. He recognized this as a Cantor set.

Mandelbrot was lucky in that he had access to the advanced computing sources at IBM, which could draw the seemingly complex transformations required by the "pre-fractals." Mandelbrot employed many techniques discovered by the early twentieth century mathematicians that had been largely forgotten by the 1970s.[12] Earlier mathematicians, like Julia, could not create images, as Mandelbrot said, "There was a long hiatus of a hundred years where drawing did not play any role in mathematics because hand and pencil and ruler were exhausted." He believed there was a decades long absence of intuition in mathematics. In 1979, while exploring the iteration of complicated processes, like equations with square roots and sines and cosines, Mandelbrot discovered that he could create one image in the complex plane that could serve as a catalog of Julia sets.[13] Mandelbrot plotted the Julia sets on graphs, and he was able to create the complex self-similar figures Julia could only imagine. By 1980, he discovered an equation that combined all the Julia sets, f(z) = z^2 + c, and when iterated, he obtained his own set, a road-map of all Julia sets- what is now known as the Mandelbrot set.



**The Mandelbrot Set, " a Road-map of the Julia Sets"**

There are those who considered the Mandelbrot set to be the most complex and beautiful object ever encountered in mathematics. It seems to be "more fractal than any fractal." Yet, its dimension is 2.

The set can be “interpreted as an illustrated encyclopedia of an infinite number of algorithms.”[14] He first thought it was one continent, a cardioid with adjoining circles, but what he thought was bits of photographic dust turned out, at successive enlargements, to be miniature continents. Only in the late 1980s was it found to that everything in the set is connected by meandering lines like cobwebs.[15] The boundary of the Mandelbrot set is “where a program spends most of its time and makes all of its compromises;” the boundary is “where points are slowest to escape the pull of the set.”[16] Every point in the complex plane is either in the Mandelbrot set or outside of it. Cataloging the different images within it or producing a numerical description of the set's outline would require an infinite amount of information.[17]

The eye, again according to Mandelbrot, had been banished out of science, and fractals brought it back in. Mandelbrot issued a bold statement about the state of mathematics and posed a challenged- mathematicians needed to rethink what mathematics could do. Initially, mathematicians did not accept Mandelbrot's work. They continued clinging to old paradigms; they were too habituated to working with smooth curves. He responded with his seminal work, *The Fractal Geometry of Nature*, in which he illustrated how fractals could give precise measurements for natural shapes and how calculations could be applied to all sorts of formations and natural systems. He proposed a new science and a new way of looking at world- a deeper way of understanding nature with mathematics.

These fractal figures have always been there, it only took the development of fractal geometry to show them- fractal geometry made the invisible, visible. Fractal geometry was indeed a new language that was dependent, not on basic shapes like circles and triangles as in Euclidean geometry, but instead upon algorithms, which could be transformed into curves, shapes, and structures only with the aid of a computer. Euclidean shapes are described by simple algebraic formulas, but fractals rely on recursive algorithms to produce them. A fractal curve “implies an organizing structure that lies hidden among the hideous complication of shapes.”[18] Iteration is key to creating fractals. When you iterate an equation instead of solving it, it becomes a process instead of a description; it is now dynamic instead of static.[19] The simplest symmetrical figure of this nature is that of a shape which repeats itself infinitely, moving in the same fixed distance and in the same fixed direction.[20] There are an infinite number of scales within a mathematical fractal. Above all fractal means self-similarity, which leads to symmetry across scale and patterns inside of patterns. According to Mandelbrot, the notion that different parts of a figure is self-similar only applies between certain limits. When each piece of a shape is geometrically similar to the whole, both the shape and the "cascade," the generating mechanism, are self-similar.[21] To obtain a Koch curve, a "cascade of smaller and smaller new promontories is pushed to infinity, but in nature, every cascade must stop or change character."[22] Translational symmetry would be defined by motion of a translation; a figure is physically translated to a new position, yet though all parts have been shifted, the appearance is unchanged.[23] Bilateral symmetry is symmetry under reflection. With fractals, though they are crinkled, fragmented, or convoluted shapes, you can zoom in and they will look pretty much the same as the picture you started with- the intricate structure remains as the figure is statistically self-similar, while if you magnify a conventional curve it will only look flatter/straighter.[24] Surprisingly, there exist lots of objects like this in mathematics. More realistic fractals require self-similarity to be interpreted statistically- each part of the fractal has the same statistical properties of form.[25]

Space is where fractals live; a fractal can be viewed as a subset of a metric space. The points in the space are elements of the set, and there is structure to the set. Metric spaces are of an "inherently simple geometric character," but their subsets can be geometrically complicated.[26] A metric space function measures the distance between a pair of points in space. A fractal set contains infinitely many points; their organization is very complex and one cannot describe it in terms of relationship between different parts. The term "fractal" is a modifier meant to exclude planes and lines. Fractal sets can be curves, surfaces, or "dusts" (point-sets). A fractal is made of an infinite number of points, and we can only see a fraction of them. To see a fractal, points can be distributed to give the illusion of seeing a complete fractal figure. The Hénon mapping model creates fractal dust. From these dust clouds, fractals are created, built up according to the principle of a binary/n-ary tree and subjected to an infinite sequence of similarity transformations (rotations, reflections). In the simple binary tree fractal, which is confined within a right triangle, as the branches get smaller they get closer to the hypotenuse of the right triangle, but they are always one branch away, and as such, there is a limit process involved.[27] Limit points are often used to describe fractals; the idea that a sequence of numbers approach a limit can be extended to a point set in a plane, which means that it can be applied to fractals.[28] Limits can be added to the dust fractal to fill in any holes; limit points can, in addition, produce a plane filling fractal.[29]

Anything fractal obeys simple rules, and a simple code can produce complexity. Fractal figures can be scaled down and rotated, like the Koch curve, Sierpinski triangle, and the Julia sets. Transformations, also known as mappings, on the plane are procedures for moving points in a plane to new locations; transformations can cause distortions and can alter shape. However, symmetries rely on transformations that leave their patterns invariant. Invariance with respect to a transformation means that if P is a point of the fractal , the point obtained by transforming P will also belong to the fractal.[30] Examples of transformation functions for basic symmetries include: $T(x,y) = (x + a, y+b)$, which moves a point a units up and b units over, $R(x,y)=(x*\cos(\theta)-y*\sin(\theta), x*\sin(\theta)+y*\cos(\theta))$, which is an example of a counterclockwise rotation about the origin through θ, and $S(x,y)=(-x,y)$, a reflection across y-axis.[31] Compositions of translations, as in S(T(x,y)), can also be created, and if S and T are symmetry transformations then so are S(T(x,y)), as well as, $S^{-1}$ and $T^{-1}$. The combination of all possible compositions is the orbit. Symmetry can also be created from maps which distort, stretch, and twist, like Möbius maps, discovered by August Möbius, on the extended complex plane, which maps circles to circles.[32]

Dimension can be subjective, as with a piece of thread- it can be viewed as a one dimensional line but it could also, if you zoom in, be seen as a thin cylinder. Fractals have fractal dimension, as fractals are “dimensionally discordant,” a concept attributed to Felix Hausdorff in 1919; the Hausdorff-

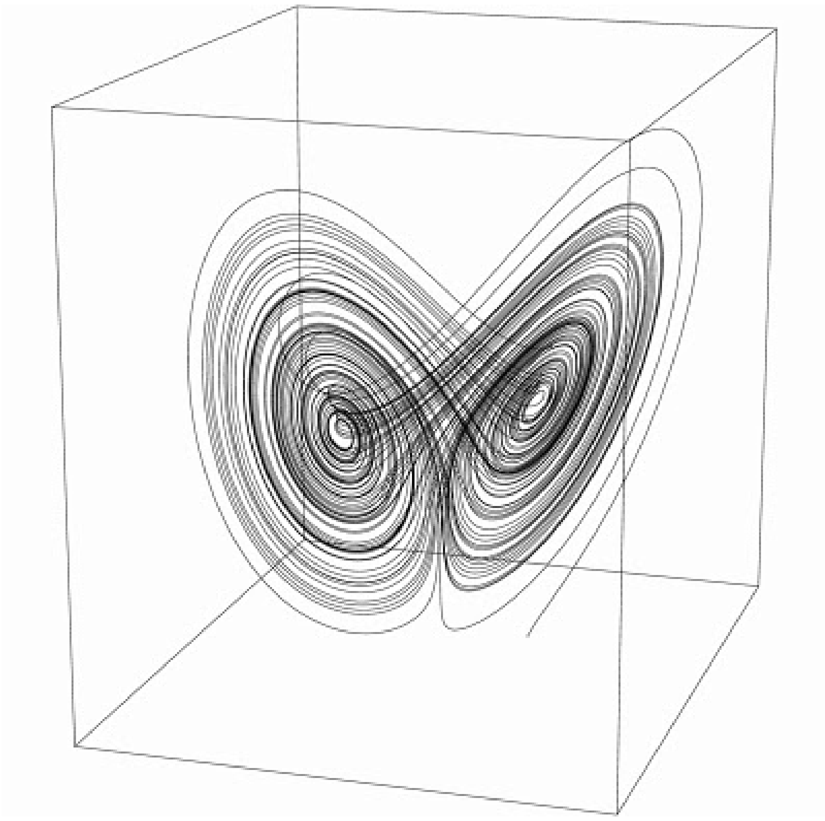
Besicovitch dimension, or the fractional dimension, relates to the "capacity" of a figure and exceeds the topological dimension.[33] He invented a way of determining a "D-dimensional measurement" for a unique number D, neither 0 nor infinity, in which D is the Hausdorff dimension.[34] The degree of irregularity of a fractal shape remains constant over all scales (regular irregularity), and the degree of irregularity corresponds to efficiency of the figure in taking up space. The Hausdorff dimension "preserves the ordinary dimension's role as an exponent in defining a measure," and curves with fractal dimensions exceeding the topological dimension of 1 are fractal curves.[35] Dimension is given by: D=log(N)/log(1/r), in which a self-similar object of N parts scaled by a ratio r. The dimension of the Koch curve would be represented by: D=log(4)/log(3), as the Koch curve composed of 4 sub-segments scaled down by 1/3. For the Koch curve, its dimension is equal to about 1.26. For another simple example, consider a line. Now cut that in half, and now cut the two halves in half. N would equal 2, and r would be ½. Thus D = 1, and of course a line is one dimensional. Fractal dimension still maintains shapes of "regular" dimension; fractal dimension agrees with our intuitive notion of dimension.[36] As D increases from 1 towards 2, the curves are less "line-like," and they start filling space.[37] A fractal with a dimension between 0 and 1, like the Cantor Set, does not contain lines and is fractal dust.[38] "A multiplicity of different dimensions is unavoidable;" however, this idea can help "transform the concept of fractal from an intuitive to a mathematical one," and differences in fractal dimension highlight differences in the non-topological aspect of form- they give a sense of the fractal form.[39] As it does not say much about what the geometric figure actually looks like, moreover, the fractal dimension can be considered to be just a "by-product."

The complexity of nature is not attributable to accidental randomness. The coexistence of both determinism and accidental development is a rule in nature. Stochastic fractals take into account "chance," and the most useful fractals do involve chance; when a small disturbance is included in the construction of a fractal, it serves as a model for natural objects. Deterministic fractal geometry concerns "subsets of a space which are generated by, or possess invariance properties under simple geometrical transformations of the space into itself."[40] In deterministic geometry, structures are "defined, communicated, and analyzed, with the aid of elementary transformations, scaling, rotations, and congruencies."[41] The Julia sets are deterministic fractals, for example. A deterministic fractal is also known as an attractor ("strange attractors," as they are also referred to, suggest that mathematicians and scientists are highly surprised by these figures).[42] An iterated function system is a dynamical system, which possess attractors, and can be described as "simply a finite set of maps acting on a complete metric space."[43] A dynamical system is a process that evolves in time, and they occur in all branches of science- for example, economics and weather; even the simplest dynamical systems which depends upon only one variable can create "highly unpredictable results and essentially random behavior," which is in fact due to "chaos."[44] A dynamical system reveals itself as a repeated transformation of a plane and leads to self-similar geometric patterns/fractals.[45] Henri Poincare discovered dynamical systems, and it can be said that he predicted fractal-like structures long before Mandelbrot.[46] Poincare mappings are iterative mappings in the plane that are measure-conserving.

Fractals can be found truly everywhere. Fractal geometry has been embraced in the applied sciences and has offered insight into a wide array of areas, like geology, particularly seismology, and biology, as there are fractals within the human body, as well as chaos theory. Mountains can be seen as endless iterations of triangular shapes. The moon's surface of craters is also fractal. Earthquakes are self-similar- large earthquakes are only scaled up versions of smaller ones. Plants have fractal structures. The pattern of branching in a single tree is self-similar. Biologists are now using a single tree to describe how an entire rainforest works; fractals have helped them understand a forest's great complexity. Mother Nature clearly has found that fractals work best in creating life. Inside the human body exist fractals, controlling many structures- the structures of our kidneys, lungs, and circulatory system (blood vessels are similar to the Koch curve and seem to perform "dimensional magic") are fractal.[47] A healthy heartbeat has a fractal pattern. Eye movement also has a fractal pattern. They have been present in art for years, for example Hokusai's "The Great Wave." Cell phones now employ the use of fractals in antenna design; a wider range of frequencies are made possible if antennas are constructed in self-similar patterns.

**“The Great Wave”**

Weather systems are fractal. Meteorologist Edward Lorenz, in 1963, produced a system of equations based on weather data and, by accident, had noticed a strange occurrence when he iterated his data set. While he could not accurately see what was actually happening on the computers available at the time, nonetheless, he had “boiled down weather to the barest skeleton” and created a tiny model of the Earth's weather.[48] The Lorenz attractor, which resembles a butterfly, never intersects with itself; it loops and spirals infinitely deep, yet stays within finite space. “Strange attractors,” like the Lorenz attractor, are created from iterative processes that are not area-conserving. His attractor is an image of “predictability giving way to pure randomness,” and it “illustrated stability and the hidden structure of a system that otherwise seemed patternless.”[49] It very much represented order disguised as randomness. Lorenz's system is of infinite complexity and highly sensitive to initial conditions. Lorenz's work also gave rise to “the butterfly effect,” the suggestion that a flap of a butterfly's wings in Africa could go on to cause a hurricane in Louisiana. The Lorenz attractor was also a stepping stone in the development of the field of chaos theory.



**The Lorenz Attractor**

Chaos theory is how simple systems give rise to complex behavior or complex systems give rise to simple behavior.[50] The development of chaos theory marked the end of reductionism in science. Everything tends towards disorder.[51] Fractals point to disorder, but scaling fractals brings in a sense of order.[52] Chaos seems to be everywhere. It is the science of the global nature of systems and the "universal behavior of complexity."[53] It can also be classified as the study of nonlinear systems, with nonlinear meaning that the "rule determining what a piece of a system is going to do next is not influenced by what it is doing now."[54]

Fractals have always existed, they were there just waiting to be discovered, but it took the advent of the computer to finally expose them. The complexity of nature and the existence of the pathological monsters made things hard for mathematicians and scientists to understand. The nineteenth century monsters were banished to a “mathematical zoo,” as mathematicians saw no use or interest in them, not until 1970s, when Mandelbrot came along and suggested that one should look at what it took to produce the figures, were they even taken seriously.[55] By 1975, fractal geometry was on its way to becoming a new and exciting field in mathematics. Fractals offer ways of seeing infinity, and we can now appreciate the potential these figures hold.[56]

**Notes**

1. David Mumford, Caroline Series, and David Wright, *Indra's Pearls: The Vision of Felix Klein* (Cambridge: Cambridge University Press, 2002), 1.

2. Ibid., 2.

3. Ibid., 4.

4. Ibid., 5.

5. James Gleick, *Chaos: Making a New Science* (New York: Penguin, 1987), 93.

6. Hans Lauwerier, *Fractals: Endlessly Repeated Geometric Figures* (Princeton, NJ: Princeton University Press, 1991), 124.

7. Ibid., 148.

8. Ibid., 54.

9. Ibid., 69.

10. Gleick, *Chaos,* 86.

11. Ibid., 86.

12. Ibid., 102.

13. Ibid., 221.

14. Hartmut Jurgens, Heinz-Otto Peitgen, and Dietmar Saupe. *Chaos and Fractals: New Frontiers of Science (*New York: Springer-Verlag, 1992).

15. Lauwerier, *Fractals*, 150.

16. Gleick, *Chaos*, 232.

17. Ibid., 221.

18. Ibid., 114.

19. Ibid., 227.

20. Mumford, *Indra's Pearls*, 5.

21. Benoit B. Mandelbrot, *The Fractal Geometry of Nature* (New York: W.H. Freeman and Company, 1977), 34.

22. Ibid., 38.

23. Mumford, *Indra's Pearls*, 5.

24. Ibid., 140.

25. Lauwerier, *Fractals*, 104.

26. Michael  Barnsley, *Fractals Everywhere* (San Diego, CA: Academic Press, 1988), 16.

27. Lauwerier, *Fractals*, 3.

28. Ibid., 26.

29. Ibid., 26.

30. Ibid., 78.

31. Mumford, *Indra's Pearls*, 7.

32. Ibid., 70.

33. Mandelbrot, *The Fractal Geometry of Nature*, 16.

34. Mumford, *Indra's Pearls*, 136.

35. Mandelbrot, *The Fractal Geometry of Nature*, 31.

36. *The Science of Fractal Images*, edited by Heinz-Otto Peitgen and Dietmar Saupe (New York: Springer-Verlag, 1988), 21.

37. Ibid., 29.

38. Lauwerier, *Fractals*, 80.

39. Mandelbrot, *The Fractal Geometry of Nature*, 17.

40. Barnsley, *Fractals Everywhere*, 43.

41. Ibid., 5.

42. Ibid., 80.

43. Ibid., 82.

44. *The Science of Fractal Images*, 138.

45. Lauwerier, *Fractals*, 125.

46. Ibid., 126.

47. Gleick, *Chaos*, 109.

48. Ibid., 115.

49. Ibid., 130.

50. Ibid., 304.

51. Ibid., 308.

52. Mandelbrot, *The Fractal Geometry of Nature*, 18.

53. Gleick, *Chaos*, 5.

54. Jurgens, *Chaos and Fractals*, 1.

55. *The Science of Fractal Images*, 25.

56. Gleick, *Chaos*, 98.

**Bibliography**


Barnsley, Michael. *Fractals Everywhere.* San Diego, CA: Academic Press, 1988.

Fractals: Hunting the Hidden Dimension. Directed by Michael Schwarz and Bill Jersey. 2008. Boston: WGBH BOSTON, 2008. DVD.

Gleick, James. *Chaos: Making a New Science*. New York: Penguin, 1987.

Jurgens, Hartmut, Heinz-Otto Peitgen, and Dietmar Saupe. *Chaos and Fractals: New Frontiers of Science.* New York: Springer-Verlag, 1992.

Lauwerier, Hans. *Fractals: Endlessly Repeated Geometric Figures*. Princeton, NJ: Princeton University Press, 1991.

Mandelbrot, Benoit B. *The Fractal Geometry of Nature*. New York: W.H. Freeman and Company, 1977.

Mumford, David, Caroline Series, and David Wright. *Indra's Pearls: The Vision of Felix Klein*. Cambridge: Cambridge University Press, 2002.

*The Science of Fractal Images.* Edited by Heinz-Otto Peitgen and Dietmar Saupe. New York: Springer-Verlag, 1988.